Constrained minimization with second order conditions

A factory produces items q_1 and q_2. The cost function is $C = q_1^2 + 2q_2^2 - q_1$ and the goal is to minimize the cost. Determine the minimum cost and the quantities to produce if the total number of items must be 8.

Solution

The constraint is:

$$q_1 + q_2 = 8$$

The Lagrangian of the problem is:

$$L = q_1^2 + 2q_2^2 - q_1 + \lambda(8 - q_1 - q_2)$$

We calculate the first-order conditions:

$$L'_{q_1} = 2q_1 - 1 - \lambda = 0$$

$$L'_{q_2} = 4q_2 - \lambda = 0$$

$$L'_{\lambda} = 8 - q_1 - q_2 = 0$$

From the first two equations, we solve for λ and set them equal:

$$2q_1 - 1 = 4q_2$$

Solving for one variable:

$$q_1 = 2q_2 + \frac{1}{2}$$

Insert into the third condition:

$$8 - (2q_2 + \frac{1}{2}) - q_2 = 0$$

$$7.5 - 3q_2 = 0$$

$$q_2 = 2.5$$

With this, we obtain the value of q_1:

$$q_1 = 2 \cdot 2.5 + \frac{1}{2} = 5.5$$

We calculate the second derivatives for the second-order conditions:

$$L''_{q_1 q_1} = 2$$

$$L''_{q_2 q_2} = 4$$

$$L''_{q_1 q_2} = L''_{q_2 q_1} = 0$$

$$g'_{q_1} = g'_{q_2} = 1$$

We form the bordered Hessian:

$$|\bar{H}| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 4 \end{vmatrix}$$

$$|\bar{H}| = -6$$

We have a minimum.